**A**nnals of **F**uzzy **M**athematics and **I**nformatics
Volume x, No. x, (mm 201y), pp. 1–xx
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

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# Partial overlap functions, continuous partial t-norms and related algebraic structures

XingNa Zhang, Nan Sheng, Rajab Ali Borzooei, EunSuk Yang, XiaoHong Zhang

Received 18 January 2025; Revised 4 February 2025; Accepted 25 March 2025

Abstract. To expand the study of undefined cases, this paper extends the aggregation operator from fuzzy logic to partial fuzzy logic. It first introduces the partial overlap function and relaxes its boundary conditions, thereby defining the concept of a partial general overlap function and outlining its construction. Additionally, this paper presents the notion of a continuous partial t-norm to explore the relations between various classes of partial operators. Furthermore, the structural features of several operators are algebraically characterized, including partial general residuated lattices, partial inflationary BL-algebras, and partial BL-algebras. The paper also investigates the properties of these operators and the relations among these algebraic structures.

2020 AMS Classification: 03B52, 47S40

Keywords: Fuzzy logic, Partial overlap function, Partial t-norm, Partial general residuated lattice.

Corresponding Author: XiaoHong Zhang (zhangxiaohong@sust.edu.cn)

## 1. Introduction

Fuzzy logic plays a crucial role in uncertainty research. Key algebraic structures, such as residuated lattices [1, 2, 3, 4, 5], which correspond to the t-norm and its residual implication, as well as algebraic structures associated with overlap functions [6, 7], have been the focus of extensive study by many scholars. In 1994, Foulis and Bennett introduced the concept of effect algebra [8], which has since been widely researched alongside residuated lattices [9, 10, 11, 12, 13]. Specifically, Zhou et al. [14] explored lattice effect algebras and residuated lattices, examining the relations between partial algebraic structures and quantum logic. As research progressed, it became apparent that fuzzy logic has certain limitations in practical applications.

To address these challenges and capture a broader range of uncertainties, Běhounek et al. have introduced partial fuzzy logic [15, 16], which effectively describes "undefined" situations (a special kind of uncertainty). They applied partial fuzzy logic to extend the ordinary discrete fuzzy transformation technique, thus generalizing it. Borzooei started with t-norm and introduced [17] an operator under partial fuzzy logic: partial t-norm, and Ji defined partial triangular implication [18]. Further studies have focused on algebraic structures related to operators in partial fuzzy logic. Zhang et al. [19] delved into partial t-norms, establishing their connection with lattice effect algebras. They also introduced partial residuated lattices and investigated their properties. T-norms have been widely applied in image processing and classification tasks, although their associativity often does not work. Overlap function, a non-associative aggregation function introduced in [20], extends the t-norm while satisfying continuity. Its key feature is that associativity is not required for information aggregation. Extensive research has been conducted on overlap functions [21, 22, 23, 24, 25, 26, 27, 28]. In particular, Paiva et al. studied the naBL algebra based on overlap function in [29], and Wang constructed overlap functions on bounded lattices [30]. These functions have found applications in various fields, including image processing, data classification, and decision-making [31, 32, 33, 34]. The general overlap function was introduced by De Miguel et al. [35] as an extension of the overlap function, differs in its relaxed boundary conditions, making its application more versatile [36, 37].

A review of the existing literature reveals that operators in partial fuzzy logic play a crucial role in research related to both partial fuzzy logic and quantum logic. Therefore, extending the current research on these operators is highly valuable. While the theoretical study of partial t-norms is relatively well-established, there is still a gap in research on overlap functions and general overlap functions within the context of partial fuzzy logic. To address this gap, we will focus on exploring operators such as partial overlap functions and partial general overlap functions, along with the related algebraic structures.

This paper is structured as such: The second section reviews the relevant existing knowledge that lays the foundation for the subsequent discussions. The third section defines the concept of partial overlap function and its induced partial residual implication in the context of partial t-norm and overlap function, exploring their properties. It then expands the concept of partial t-norm, introducing the new notion of continuous partial t-norm and investigating its related properties. Next, the concept of partial general overlap function is introduced by relaxing the boundary conditions of partial overlap function, and construction method is explored. The relations among these operators are examined with continuous partial t-norm acting as a connecting element. The fourth section focuses on the algebraic aspects of the operators discussed. It begins by introducing the concept of partial general residuated lattice and proving that the partial algebraic structure corresponding to partial general overlap function and its derived partial residual implication forms a partial general residuated lattice. In fuzzy logic, each continuous t-norm on [0, 1] corresponds to a BL-algebra, which was introduced by Hájek in [38, 39]. Based on this, We define a partial BL-algebra, which consists of a continuous partial t-norm and its derived residual implication. Specifically, we introduce the new concept of partial inflationary BL-algebra and prove that the partial algebraic structure corresponding to the inflationary partial overlap function and its derived partial residual implication forms a partial inflationary BL-algebra. The relations among these partial algebraic structures are also analyzed. The final section summarizes the findings of the research and outlines directions for future work.

## 2. Preliminaries

In this section, we will give some existing conclusions to facilitate their application in subsequent sections.

**Definition 2.1** ([20])**.** Let $\otimes$ be an operator on $[0,1]$. If it meets the terms below: for all $s,t,m \in [0,1]$,
(i) $s \otimes t = t \otimes s$,
(ii) $st = 0$ iff $s \otimes t = 0$,
(iii) $st = 1$ iff $s \otimes t = 1$,
(iv) if $t \geq m$, then $s \otimes m \leq s \otimes t$,
(v) $\otimes$ is continuous,
then $\otimes$ is referred to as an *overlap function*.

**Definition 2.2** ([20])**.** Let $\otimes$ be an operator on $[0,1]$. If it meets the terms below: for all $s,t,m \in [0,1]$,
(i) $s \otimes t = t \otimes s$,
(ii) if $st = 0$, then $s \otimes t = 0$,
(iii) if $st = 1$, then $s \otimes t = 1$,
(iv) if $t \geq m$, then $s \otimes m \leq s \otimes t$,
(v) $\otimes$ is continuous,
then $\otimes$ is referred to as a *general overlap function*.

**Definition 2.3** ([40])**.** Let $\otimes$ be a general overlap function on [0, 1]. Then $\otimes$ is said to be $inflationary$, if $s \otimes 1 \geq s$ for each $s \in [0,1]$.

**Definition 2.4** ([17])**.** Let $A$ be a bounded lattice and $\otimes$ be an operator on $A$. If it meets the conditions below: for all $s,t,m \in A$,
(i) $s \otimes 1 = s$,
(ii) if $t \otimes s$ is defined, then $s \otimes t$ is also defined, and $t \otimes s = s \otimes t$,
(iii) if $t \otimes m$ and $s \otimes (t \otimes m)$ are defined, then $s \otimes t$ and $(s \otimes t) \otimes m$ are also defined and $s \otimes (t \otimes m) = (s \otimes t) \otimes m$,
(iv) if $s \leq t$, $m \leq n$, and $s \otimes m$, $t \otimes n$ are defined, then $s \otimes m \leq t \otimes n$,
then $\otimes$ is referred to as a *partial t-norm*.

**Definition 2.5** ([1])**.** A structure $(A, \wedge, \vee, \otimes, \rightarrow_{\otimes}, 0, 1)$ is referred to as a *residuated lattice*, if the terms below are met:
(i) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice, where 0 is the minimum and 1 is the maximum of $A$, respectively,
(ii) $(A; \otimes, 1)$ is a commutative monoid,
(iii) $(\otimes, \rightarrow_{\otimes})$ is an adjoint pair on $A$, which $m \leq s \rightarrow_{\otimes} t$ if and only if $s \otimes m \leq t$ for all $s,t,m \in A$.

**Definition 2.6** ([19]). Let $\otimes$ and $\rightarrow_\otimes$ be two binary operators, and ($A$, $\wedge$, $\vee$, 0, 1) is a bounded lattice. The algebra ($A$, $\wedge$, $\vee$, $\otimes$, $\rightarrow_\otimes$, 0, 1) is said to be a *partial residuated lattice* (PRL), provided that for all $s, t, m \in A$,

(i) if $t \otimes s$ is defined, then $s \otimes t$ is defined, $t \otimes s = s \otimes t$,

(ii) if $t \otimes m$, $s \otimes (t \otimes m)$ are defined, then $s \otimes t$ and $(s \otimes t) \otimes m$ are defined, and $(s \otimes t) \otimes m = s \otimes (t \otimes m)$,

(iii) $s \otimes 1 = s$,

(iv) ($\otimes$, $\rightarrow_\otimes$) a partial adjoint pair on $A$.

**Definition 2.7** ([40]). Let $\otimes$: $A^2 \rightarrow A$ and $\rightarrow_\otimes$: $A^2 \rightarrow A$ be two binary operators on $A$. ($A$, $\wedge$, $\vee$, $\otimes$, $\rightarrow_\otimes$, 0, 1) is referred to as a *general residuated lattice* (GRL), if it meets the terms below:

(i) ($A$, $\wedge$, $\vee$, 0, 1) is a bounded lattice with maximum 1 and minimum 0 values,

(ii) ($A$, $\otimes$) is a commutative groupoid,

(iii) $s \otimes t \leq m$ iff $t \leq s \rightarrow_\otimes m$ for all $s, t, m \in A$.

**Remark 2.8.** The groupoid is an algebraic structure ($A$, $\otimes$), where $\otimes$ is a binary operator defined on the nonempty set $A$.

**Definition 2.9** ([40]). An algebra ($A$, $\wedge$, $\vee$, $\otimes$, $\rightarrow_\otimes$, 0, 1) is referred to as an *inflationary BL-algebra*, if it is an inflationary GRL and meets the terms below: for all $s, t \in A$,

(i) (Divisibility) $s \otimes (s \rightarrow_\otimes t) = s \wedge t$,

(ii) (General prelinearity) $(s \rightarrow_\otimes (t \otimes 1)) \vee (t \rightarrow_\otimes (s \otimes 1)) = 1$.

**Definition 2.10** ([19]). Let $A$ be a bounded lattice, where 0 is the minimum and 1 is the maximum of $A$. The binary operator $PI$: $A^2 \rightarrow A$ is referred to as a *partial fuzzy implication* (PFI), provided that for all $s, t, s_1, s_2, t_1, t_2 \in A$,

(PI1) if $s_1 \leq s_2$, $PI(s_1, t)$ and $PI(s_2, t)$ are defined, then $PI(s_2, t) \leq PI(s_1, t)$,

(PI2) if $t_1 \leq t_2$, $PI(s, t_1)$ and $PI(s, t_2)$ are defined, then $PI(s, t_1) \leq PI(s, t_2)$,

(PI3) $PI(1, 0) = 0$, $PI(0, 0) = PI(1, 1) = 1$.

**Definition 2.11** ([23]). A fuzzy implication $I : [0, 1]^2 \rightarrow [0, 1]$ meets: for all $s, t, m \in [0, 1]$,

(NP) the left neutrality property iff $I(1, s) = s$,

(EP) the exchange principle iff $I(s, I(t, m)) = I(t, I(s, m))$,

(IP) the identity principle iff $I(s, s) = 1$,

(LOP) the left ordering property iff $s \leq t \Rightarrow I(s, t) = 1$,

(ROP) the right ordering property iff $I(s, t) = 1 \Rightarrow s \leq t$,

(OP) the ordering property iff $I(s, t) = 1 \Leftrightarrow s \leq t$,

(CB) the consequent boundary iff $t \leq I(s, t)$,

(LBC) the left boundary condition iff $I(0, s) = 1$,

(RBC) the right boundary condition iff $I(s, 1) = 1$.

## 3. Partial overlap functions

The concepts of partial overlap function, partial general overlap function, and continuous partial t-norm on [0, 1] are reasonably well defined.

**Definition 3.1.** An operator $\otimes : [0,1]^2 \to [0,1]$ is called a *partial overlap function* (PO), provided that it meets the terms below: for all $s,t,m \in [0,1]$,

(i) if $s \otimes t$ and $t \otimes s$ are defined, then $s \otimes t = t \otimes s$,

(ii) if $s \otimes t$ is defined, then $s \otimes t = 0$ iff $st = 0$,

(iii) if $s \otimes t$ is defined, then $s \otimes t = 1$ iff $st = 1$,

(iv) if $s \otimes t$ and $s \otimes m$ are defined and $t \leq m$, then $s \otimes t \leq s \otimes m$,

(v) $\otimes$ is P-continuous. i.e., for all $t \in [0,1]$, unary function $h(s) = s \otimes t (s \in [0,1])$ meet the terms: for all $s_1, s_2, p \in [0,1]$, if $h(s_1), h(s_2)$ are defined and $h(s_1) \leq p \leq h(s_2)$, then there exists $u \in [0,1]$ that holds $h(u) = p$.

**Definition 3.2.** Let $\otimes$ be a PO on [0, 1]. Then $\to_\otimes$ is the operator defined as below: for all $s,t,m \in [0,1]$,

$$s \to_\otimes t := \begin{cases} \sup\{m | s \otimes m \text{ is defined and } s \otimes m \leq t\}, & M \neq \varnothing \text{ and } \sup M \text{ exists} \\ \text{undefined}, & \text{otherwise} \end{cases}$$

Particularly, $M = \{m | s \otimes m \text{ is defined and } s \otimes m \leq t\}$. Then binary operator $\to_\otimes$ is called the partial residual implication (PRI) induced by $\otimes$.

**Example 3.3.** The binary operator $\otimes$ is a PO and $\to_\otimes$ is the PRI that is induced by $\otimes$ below $(s,t \in [0,1])$:

$$s \otimes t := \begin{cases} s^2t^2 & \text{otherwise} \\ \text{undefined} & \text{if } s,t \in [0,0.4], \end{cases}$$

$$s \to_\otimes t := \begin{cases} \frac{\sqrt{t}}{s} & \text{otherwise} \\ 1 & \text{if } t \geq s^2 \\ \text{undefined} & \text{if } s,t \in [0,0.4] \text{ and } t < s^2. \end{cases}$$

**Theorem 3.4.** *Let $\otimes$ be a PO on $[0,1]$ and $\to_\otimes$ be the PRI that is derived from $\otimes$. Then $\to_\otimes$ is a PFI.*

*Proof.* (PI1) For all $s,t,m,a,b \in [0,1]$, if $t \to_\otimes m$ and $s \to_\otimes m$ are defined one gets

$$s \to_\otimes m = \sup\{a | s \otimes a \text{ is defined and } s \otimes a \leq m\}$$

and

$$t \to_\otimes m = \sup\{b | t \otimes b \text{ is defined and } t \otimes b \leq m\},$$

then there exists $b$ that holds $t \otimes b$ is defined and $t \otimes b \leq m$. Thus $b \to_\otimes m \geq t$. And if $s \leq t$, then $s \leq b \to_\otimes m$. Thus $s \otimes b$ is defined and $s \otimes b \leq m$. So we have

$$b \in \{a | s \otimes a \text{ is defined and } s \otimes a \leq m\}$$

and

$$\{b | t \otimes b \text{ is defined and } t \otimes b \leq m\} \subseteq \{a | s \otimes a \text{ is defined and } s \otimes a \leq m\}.$$

Hence $\sup\{b | t \otimes b \text{ is defined and } t \otimes b \leq m\} \subseteq \sup\{a | s \otimes a \text{ is defined and } s \otimes a \leq m\}$. Therefore we can get $s \to_\otimes m \geq t \to_\otimes m$.

(PI2) For all $s,t,m,c,d \in [0,1]$, if $s \to_\otimes t$ and $s \to_\otimes m$ are defined one gets

$$s \to_\otimes t = \sup\{c | s \otimes c \text{ is defined and } s \otimes c \leq t\},$$

$$s \to_\otimes m = \sup\{d | s \otimes d \text{ is defined and } s \otimes d \leq m\},$$

then there exists $c$ that holds $s \otimes c$ is defined and $s \otimes c \leq t$. And if $t \leq m$, then $s \otimes c \leq m$. Thus $c \in \{d|s \otimes d$ is defined and $s \otimes d \leq m\}$. So we have

$$\{c|s \otimes c \text{ is defined and } s \otimes c \leq t\} \subseteq \{d|s \otimes d \text{ is defined and } s \otimes d \leq m\}.$$

It is clearly that

$$\sup\{c|s \otimes c \text{ is defined and } s \otimes c \leq t\} \subseteq \sup\{d|s \otimes d \text{ is defined and } s \otimes d \leq m\}.$$

Hence we can obtain $s \to_\otimes t \leq s \to_\otimes m$.

(PI3) For all $m \in [0,1]$, if $0 \otimes 1$ and $1 \otimes 1$ are defined and

$$\begin{aligned} 0 \to_\otimes 0 &= \sup\{m|0 \otimes m \text{ is defined and } 0 \otimes m \leq 0\} \\ &= \sup\{m|0 \otimes m \text{ is defined and } 0 \otimes m = 0\} \\ &= 1, \end{aligned}$$

then $0 \to_\otimes 0 = 1$, $1 \to_\otimes 1 = \sup\ \{m|1 \otimes m$ is defined and $1 \otimes m \leq 1\} = 1$. Thus $1 \to_\otimes 1 = 1$, $1 \to_\otimes 0 = \sup\ \{m|1 \otimes m$ is defined and $1 \otimes m \leq 0\} = \sup\ \{m|1 \otimes m$ is defined and $1 \otimes m = 0\} = 0$. So $1 \to_\otimes 0 = 0$. □

**Proposition 3.5.** *Let $\otimes$ be a PO on $[0,1]$, then the cases below are valid: for all $s, t, n, m \in [0,1]$,*

(1) *if $1 \otimes s$ is defined, then $\to_\otimes$ meets (NP) iff 1 is the unit element of $\otimes$,*

(2) *if $t \otimes m, s \otimes m, s \otimes (t \otimes m)$ and $t \otimes (s \otimes m)$ are defined, then $\to_\otimes$ satisfies (EP) iff $\otimes$ is associative, $s \otimes (t \otimes m) = (s \otimes t) \otimes m$,*

(3) *if $s \otimes 1$ is defined, then $\to_\otimes$ satisfies (IP) iff $s \otimes 1 \leq s$,*

(4) *if $1 \otimes s$ and $s \otimes m$ are defined, then $\to_\otimes$ satisfies (LOP) iff $s \otimes 1 \leq s$,*

(5) *if $1 \otimes s$ is defined, then $\to_\otimes$ satisfies (ROP) iff $s \otimes 1 \geq s$,*

(6) *if $1 \otimes s$ and $s \otimes m$ are defined, then $\to_\otimes$ satisfies (OP) iff $s \otimes 1 = s$,*

(7) *if $s \otimes t$ and $s \to_\otimes min\{s,t\}$ are defined and $s \otimes t \leq min\{s,t\}$, then $\to_\otimes$ satisfies (CB),*

(8) *if $0 \otimes m$ is defined, then $\to_\otimes$ satisfies (LBC) and (RBC),*

(9) *if $s \otimes m, s \otimes 0$ and $s \otimes t$ are defined and $s \neq 1$, then $s \otimes m = s \otimes t$ iff $s = 0$ or $t = m$.*

*Proof.* (1) ($\Rightarrow$) Suppose the sufficient condition holds and let $s, t, n, m \in [0,1]$. If for all $s \in [0,1], 1 \to_\otimes s = \sup\ \{m|1 \otimes m$ is defined and $1 \otimes m \leq s\} = s$. Then $1 \otimes s \leq s$. Assume that for $s_1 \in [0,1], 1 \otimes s_1 < s_1$ and let $n = 1 \otimes s_1$. Then $n < s_1 \leq 1 \to_\otimes n = n$, but there is incompatible. Thus $1 \otimes s_1 \geq s_1$. To sum up, $1 \otimes s = s$ and 1 is the unit element of $\otimes$.

($\Leftarrow$) Suppose the necessary condition holds, i.e., $1 \otimes s = s$ for all $s \in [0,1]$ and, then $\sup\ \{m|1 \otimes m$ is defined and $1 \otimes m \leq s\} = \sup\ \{m|m \leq s\} = s$.

(2) Suppose the necessary condition holds and assume that $s \otimes (t \otimes m) = t \otimes (s \otimes m) = n$. Not hard to know that $t \otimes m = s \to_\otimes n$ and $s \otimes m = t \to_\otimes n$. Then $m = t \to_\otimes (s \to_\otimes n)$ and $m = s \to_\otimes (t \to_\otimes n)$. Thus $t \to_\otimes (s \to_\otimes n) = s \to_\otimes (t \to_\otimes n)$. The reverse is also true.

(3) Suppose $1 \otimes s$ is defined. Then

$$\begin{aligned} & s \to_\otimes s = \sup\ \{m|s \otimes m \text{ is defined and } s \otimes m \leq s\} = 1 \\ \Leftrightarrow\ & s \otimes 1 \leq s. \end{aligned}$$

(4) ($\Rightarrow$) Suppose the sufficient condition holds. Since $s \to_\otimes s = \sup\ \{m|s \otimes m$ is defined and $s \otimes m \leq s\} = 1$, one gets $s \otimes 1 \leq s$.

($\Leftarrow$) When $s \leq t$, it is clear that $s \otimes m \leq s \otimes 1 \leq s \leq t$, then $s \rightarrow_{\otimes} t = \sup \{m | s \otimes m$ is defined and $s \otimes m \leq t\} = 1$.

(5) ($\Rightarrow$) Suppose the sufficient condition holds. Then we have

$$s \rightarrow_{\otimes} (s \otimes 1) = \sup\{m | s \otimes m \text{ is defined and } s \otimes m \leq s \otimes 1\} = 1.$$

Thus $s \otimes 1 \geq s$.

($\Leftarrow$) Suppose $s \otimes 1 \geq s$. Since $s \rightarrow_{\otimes} t = \sup \{m | s \otimes m$ is defined and $s \otimes m \leq t\} = 1$, one gets $s \otimes 1 \leq t$, farther, $s \leq s \otimes 1 \leq t$. Thus $s \leq t$. So $\rightarrow_{\otimes}$ satisfies (ROP).

(6) This property can be easily proved to hold by (4) and (5).

(7) Suppose $s \otimes t, s \rightarrow_{\otimes} min\{s,t\}$ are defined and $s \otimes t \leq min\{s,t\} \leq t$. Then one gets

$$s \rightarrow_{\otimes} min\{s,t\} = \sup\{t | s \otimes t \text{ is defined and } s \otimes t \leq min\{s,t\}\} \geq t.$$

Since $\rightarrow_{\otimes}$ is monotonically increasing with respect to the second variable, $s \rightarrow_{\otimes} t \geq s \rightarrow_{\otimes} min\{s,t\} \geq t$.

(8) Clearly established.

(9) ($\Leftarrow$) Suppose $s = 0$ or $t = m$ and $s \otimes m, t \otimes 0, m \otimes 0$ and $s \otimes t$ are defined and $s \neq 1$. Then $0 \otimes m = 0 = 0 \otimes t$ and $s \otimes m = s \otimes t$.

($\Rightarrow$) Suppose the sufficient condition holds. Then due to the monotonicity of $\otimes$, there are three cases that hold true.

Case 1: If $0 \otimes m = 0 = 0 \otimes t$. Then $s = 0$.

Case 2: If $m = t$, then $s \otimes m = s \otimes t$.

Case 3: If $s \otimes m = s \otimes t = 1$, then it is not valid clearly.

Thus $s = 0$ or $t = m$. □

**Theorem 3.6.** *Let $\otimes$ be a PO on $[0,1]$ and $\rightarrow_{\otimes}$ be the PRI that is induced by $\otimes$. The statements below are equivalent: for any $s,t,m \in [0,1]$,*

*(1) if $\otimes$ is infinitely $\vee$-distributive, i.e., $\bigvee_{i\in I} m_i$ and $\bigvee_{i\in I}(s \otimes m_i)$ exist, then $s \otimes (\bigvee_{i\in I} m_i) = \bigvee_{i\in I}(s \otimes m_i)$,*

*(2) $s \otimes m$ is defined and $s \otimes m \leq t$ iff $s \rightarrow_{\otimes} t$ is defined and $m \leq s \rightarrow_{\otimes} t$,*

*(3) if $s \otimes m$ and $s \otimes (s \rightarrow_{\otimes} t)$ are defined, then $s \otimes (s \rightarrow_{\otimes} t) \leq t$,*

*(4) if $\{m | s \otimes m$ is defined and $s \otimes m \leq t\}$ is non-empty, then there exists a maximum of the set.*

*Proof.* (1) $\Rightarrow$ (2) For any $n \in [0,1]$, suppose that $s \otimes m$ is defined and $t \geq s \otimes m$. Then

$$m \in \{n | s \otimes n \text{ is defined and } s \otimes n \leq t\},$$

$$m \leq \sup\{n | s \otimes n \text{ is defined and } s \otimes n \leq t\} = s \rightarrow_{\otimes} t.$$

Contrarily, if $m \leq s \rightarrow_{\otimes} t$ and based on the monotonicity of $\otimes$, the we have

$$\begin{aligned} s \otimes m &\leq s \otimes (s \rightarrow_{\otimes} t) \\ &= s \otimes (\sup\{n | s \otimes n \text{ is defined and } s \otimes n \leq t\}) \\ &= \sup\{s \otimes n | s \otimes n \text{ is defined and } s \otimes n \leq t\} \\ &= t. \end{aligned}$$

Thus $s \otimes m$ is defined and $s \otimes m \leq t$.

(2)$\Rightarrow$ (3) It is obvious that $s \rightarrow_{\otimes} t \leq s \rightarrow_{\otimes} t$ and based on the residuation principle, one gets $s \otimes (s \rightarrow_{\otimes} t) \leq t$.

(3)⇒ (4) Suppose (3) holds and $\{m|s\otimes m$ is defined and $s\otimes m\leq t\}$ is not empty. Then $s\rightarrow_{\otimes} t\in\{m|s\otimes m$ is defined and $s\otimes m\leq t$ and $s\rightarrow_{\otimes} t=\sup\ \{m|s\otimes m$ is defined and $s\otimes m\leq t\}$. Thus this set has the maximum element.

(4)⇒ (1) Based on the monotonicity of $\otimes$, we get $s\otimes(\bigvee_{i\in I}m_i)\geq\bigvee_{i\in I}(s\otimes m_i)$. We prove $s\otimes(\bigvee_{i\in I}m_i)\leq\bigvee_{i\in I}(s\otimes m_i)$. Let $t=\bigvee_{i\in I}(s\otimes m_i)$. Then $s\otimes m_i\leq t$. For all $m_i\in[0,1]$, $m_i\in\{n|s\otimes n$ is defined and $s\otimes n\leq t\}$. Thus one gets $m_i\leq s\rightarrow_{\otimes} t$ and $\bigvee_{i\in I}m_i\leq s\rightarrow_{\otimes} t$. It is clear that

$$s\otimes(\bigvee_{i\in I}m_i)\leq s\otimes(s\rightarrow_{\otimes} t)\leq t=\bigvee_{i\in I}(s\otimes m_i).$$

So $s\otimes(\vee_{i\in I}m_i)=\vee_{i\in I}(s\otimes m_i)$. □

**Definition 3.7.** The binary operator $\otimes:[0,1]^2\rightarrow[0,1]$ is said to be a *continuous partial t-norm* (cp-t-norm), if it satisfies the following conditions:

(i) $\otimes$ is a partial t-norm,

(ii) $\otimes$ is P-continuous. i.e., for any $t\in[0,1]$, unary function $h(s)=s\otimes t(s\in[0,1])$ meet the terms: for all $s_1,s_2,p\in[0,1]$, if $h(s_1),h(s_2)$ are defined and $h(s_1)\leq p\leq h(s_2)$, then there exists $u\in[0,1]$ that holds $h(u)=p$.

**Example 3.8.** The binary operator $\otimes$ is showed as below: for all $s,t\in[0,1]$,

$$s\otimes t:=\begin{cases}\max\{0,s+t-1\} & \text{if } s,t\in[0,0.75] \text{ or } s=1 \text{ or } t=1\\ \text{undefined} & \text{otherwise.}\end{cases}$$

Plainly, $\otimes$ is not a cp-t-norm (Let $t=0.4,s_1=1$ and $s_2=0.5$. Then $h(s_1)=s_1\otimes t$ $=\max\ \{0,1+0.4-1\}=0.4$, $h(s_2)=s_2\otimes t=\max\ \{0,0.5+0.4-1\}=0$. Thus we can not find an $s_3$, so that $h(s_3)$ is defined and $h(s_3)=0.3\in[0,0.4]$. Hence $\otimes$ does not satisfy P-continuous).

**Example 3.9.** The binary operator $\otimes$ is a cp-t-norm and $\rightarrow_{\otimes}$ is the PRI induced by $\otimes$ below: for all $s,t\in[0,1]$,

$$s\otimes t:=\begin{cases}\min\{s,t\} & \text{otherwise}\\ \text{undefined} & \text{if } s,t\in[0,0.5],\end{cases}$$

$$s\rightarrow_{\otimes}t:=\begin{cases}\text{undefined} & \text{if } s,t\in[0,0.5] \text{ and } s>t\\ 1 & \text{if } s\leq t\\ t & \text{otherwise.}\end{cases}$$

**Theorem 3.10.** *Let $\otimes$ be a cp-t-norm, $\rightarrow_{\otimes}$ be the PRI that is induced by $\otimes$. For all $s,t\in[0,1]$, if $s\leq t,t\rightarrow_{\otimes} s$ and $t\otimes(t\rightarrow_{\otimes} s)$ are defined, then $s=t\otimes(t\rightarrow_{\otimes} s)$.*

*Proof.* For any $s,t\in[0,1],t\rightarrow_{\otimes} s$ is defined, we can get $t\rightarrow_{\otimes} s=\sup\ \{m\in[0,1]|t\otimes m$ is defined and $t\otimes m\leq s\}=\max\ \{m\in[0,1]|t\otimes m$ is defined and $t\otimes m\leq s\}$, and $t\otimes m\leq s\leq t=t\otimes 1$. By Definition 3.7, $n$ exists and $n\in[t\rightarrow_{\otimes} s,1]$ such that $t\otimes n=s$. Thus $t\otimes n\leq s$ implies $n\leq t\rightarrow_{\otimes} s$. Since $n\in[t\rightarrow_{\otimes} s,1]$, $n\geq t\rightarrow_{\otimes} s$. So we have $n=t\rightarrow_{\otimes} s$. Hence $s=t\otimes n=t\otimes(t\rightarrow_{\otimes} s)$. □

By weakening the boundary conditions of PO and requiring only necessary conditions, a new concept of partial general overlap function can be obtained.

**Definition 3.11.** An operator $\otimes : [0,1]^2 \to [0,1]$ is said to be a *partial general overlap function* (PGO), if it meets some terms below: for any $s, t, m \in [0,1]$,
(i) if $s \otimes t$ and $t \otimes s$ are defined, then $s \otimes t = t \otimes s$,
(ii) if $s \otimes t$ is defined and $st = 0$, then $s \otimes t = 0$,
(iii) if $s \otimes t$ is defined and $st = 1$, then $s \otimes t = 1$,
(iv) if $s \otimes t$ and $s \otimes m$ are defined and $t \leq m$, then $s \otimes t \leq s \otimes m$,
(v) $\otimes$ is P-continuous, i.e., for all $t \in [0,1]$, unary function $h(s) = s \otimes t$ meet the following terms: for all $s_1, s_2, p \in [0,1]$, if $h(s_1), h(s_2)$ are defined, and $h(s_1) \leq p \leq h(s_2)$, then there exists $u \in [0,1]$ that holds $h(u) = p$.

**Remark 3.12.** If $\otimes$ is a PO, then $\otimes$ is a PGO.

**Example 3.13.** Define the operators $\otimes$ and $\to_\otimes$ below: for all $s, t \in [0,1]$,

$$s \otimes t := \begin{cases} \max\{0, s^2 + t^2 - 1\} & \text{otherwise} \\ \text{undefined} & \text{if } s, t \in [0, 0.5], \end{cases}$$

$$s \to_\otimes t := \begin{cases} \sqrt{t + 1 - s^2} & \text{if } t < s^2 \\ 1 & \text{otherwise.} \end{cases}$$

Then $\otimes$ is a PGO, $\to_\otimes$ is the PRI induced by $\otimes$.

The following gives a new method of constructing a PGO.

**Definition 3.14.** The mapping $\otimes$ is shown as a PGO, if there exists binary operators $f, k : [0,1]^2 \to [0,1]$, meanwhile, both $f$ and $k$ are defined and the defined intervals have overlapping parts.

$$s \otimes t = \frac{f(s,t)}{f(s,t) + k(s,t)}$$

meets the following items: for all $s, t \in [0,1]$,
(i) $f$ and $k$ are commutative,
(ii) $f$ is increasing and $k$ is decreasing,
(iii) if $f(s,t)$ is defined and $st = 0$, then $f(s,t) = 0$,
(iv) if $k(s,t)$ is defined and $st = 1$, then $k(s,t) = 0$,
(v) $f$ and $k$ are P-continuous,
(vi) $f(s,t) + k(s,t) \neq 0$.

*Proof.* From (i), we get $f(s,t) = f(t,s), k(s,t) = k(t,s)$. Then

$$s \otimes t = \frac{f(s,t)}{f(s,t) + k(s,t)} = \frac{f(t,s)}{f(t,s) + k(t,s)} = t \otimes s,$$

$\otimes$ fulfils symmetry.

From (ii), if $s_1 \leq s_2$, then $f(s_2,t)k(s_1,t) \geq f(s_1,t)k(s_2,t)$. And we get

$$f(s_1,t)(f(s_2,t) + k(s_2,t)) \leq f(s_2,t)(f(s_1,t) + k(s_1,t)).$$

Thus

$$\frac{f(s_1,t)}{f(s_1,t) + k(s_1,t)} \leq \frac{f(s_2,t)}{f(s_2,t) + k(s_2,t)}.$$

Clearly, $s_1 \otimes t \leq s_2 \otimes t$. So $\otimes$ is increasing.

From (iii), if $f(s,t)$ is defined and $st = 0$, then $f(s,t) = 0$. Thus $s \otimes t = 0$.

From (iv), if $k(s,t)$ is defined and $st = 1$, then $k(s,t) = 0$. Thus $s \otimes t = 1$.

From (v) and (vi), it is clearly that $\otimes$ is P-continuous. In summary, $\otimes$ is a PGO. □

**Example 3.15.** Given $f(s,t)$ and $k(s,t)$ satisfying the conditions in Definition 3.14, construct PGO $\otimes$ as follows,

$$f(s,t) := \begin{cases} \max\{0, s+t-1\} \times (\min\{s,t\})^2 & \text{if } s,t \in [0.4, 0.9] \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$k(s,t) := \begin{cases} 1-st & \text{if } s,t \in [0.3, 1] \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$s \otimes t := \frac{f(s,t)}{f(s,t)+k(s,t)} = \begin{cases} \frac{\max\{0,s+t-1\}\times(\min\{s,t\})^2}{\max\{0,s+t-1\}\times(\min\{s,t\})^2+1-st} & \text{if } s,t \in [0.4, 0.9] \\ \text{undefined} & \text{otherwise} \end{cases}$$

Discuss the relations between the relevant operators studied in this paper (including PO, cp-t-norm, and well-known operators) as follows (Refer to Figure 1).

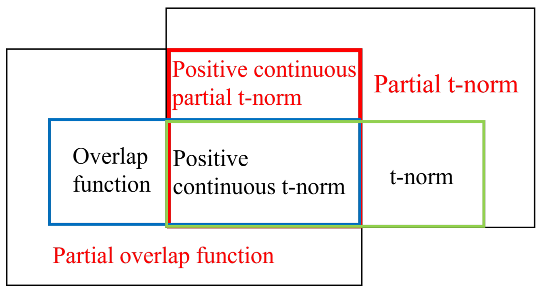


FIGURE 1. Diagram of the relations between operators such as PO, cp-t-norm, etc.

## 4. PARTIAL GENERAL RESIDUATED LATTICES

This section focuses on the study of algebraic systems such as partial general residuated lattices, partial BL-algebras and partial inflationary BL-algebras, and elaborates on the relations between each algebraic structure.

**Definition 4.1.** Let $A$ be a bounded lattice, $\otimes : A^2 \to A$ and $\to_\otimes$: $A^2 \to A$ be binary operators. Then $(A, \wedge, \vee, \otimes, \to_\otimes, 0, 1)$ is called a *partial general residuated lattice* (PGRL), if it meets the terms below: for all $s, t, m \in A$,

(i) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice with maximum 1 and minimum 0 values,
(ii) $(A, \otimes)$ is a commutative groupoid,
(iii) if $s \otimes t$ and $s \to_\otimes m$ are defined, then $m \geq s \otimes t$ iff $t \leq s \to_\otimes m$.

**Example 4.2.** Let $A = \{0, s, t, 1\}$ and Figure 2 be the diagram of $(A; \leq)$. Consider operations of $\otimes$ and $\to_\otimes$ given as Tables 1 and 2 below. Then $(A, \wedge, \vee, \otimes, \to_\otimes, 0, 1)$ is a PGRL.

**Theorem 4.3.** *Let $A = [0,1]$, $\otimes$ be a PGO and $\to_\otimes$ be the PRI induced by $\otimes$. Then $(A, \wedge, \vee, \otimes, \to_\otimes, 0, 1)$ is a PGRL.*

*Proof.* Obviously established by Definition 4.1. □

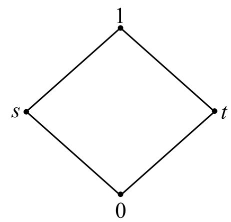


FIGURE 2. The Hasse diagram of $(A; \leq)$.

| $\otimes$ | 0 | $s$ | $t$ | 1 |
|---|---|---|---|---|
| 0 | 0 | 0 | | 0 |
| $s$ | 0 | | | |
| $t$ | | | $t$ | $t$ |
| 1 | 0 | | $t$ | 1 |

TABLE 1. Partial operation of $\otimes$.

| $\rightarrow_{\otimes}$ | 0 | $s$ | $t$ | 1 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| $s$ | $s$ | 1 | 1 | 1 |
| $t$ | 0 | $s$ | 1 | 1 |
| 1 | 0 | | $t$ | 1 |

TABLE 2. Partial operation of $\rightarrow_{\otimes}$.

**Theorem 4.4.** *Let $(A, \wedge, \vee, \otimes, \rightarrow_{\otimes}, 0, 1)$ be a PGRL. Suppose $(A, \wedge, \vee, 0, 1)$ is a complete lattice, $s \otimes m$ and $s \otimes (s \rightarrow_{\otimes} t)$ are defined and $\{m | s \otimes m$ is defined and $s \otimes m \leq t\}$ is not empty. Then for all $s, t, m \in A$,*

$$s \rightarrow_{\otimes} t = \max\{m | s \otimes m \text{ is defined and } s \otimes m \leq t\}.$$

*Proof.* If $s \otimes m$ and $s \otimes (s \rightarrow_{\otimes} t)$ are defined. It is easy to know that when $s \otimes (s \rightarrow_{\otimes} t) \leq t$, then $s \rightarrow_{\otimes} t \in \{m | s \otimes m$ is defined and $s \otimes m \leq t\}$. Since $s \rightarrow_{\otimes} t =$ sup $\{m | s \otimes m$ is defined and $s \otimes m \leq t\}$, we get

$$s \rightarrow_{\otimes} t = \max\{m | s \otimes m \text{ is defined and } s \otimes m \leq t\}.$$

□

In this part, a partial algebraic structure consisting of cp-t-norm and its PRI is presented - partial BL-algebra.

**Definition 4.5.** A PRL $\mathbf{A} = (A, \wedge, \vee, \otimes, \rightarrow_{\otimes}, 0, 1)$ is said to be a *partial BL-algebra*, if it satisfies (pbl1) and (pbl2): for all $s, t \in A$,
(pbl1) $s \rightarrow_{\otimes} t$ and $t \rightarrow_{\otimes} s$ are defined, imply $(t \rightarrow_{\otimes} s) \vee (s \rightarrow_{\otimes} t) = 1$,
(pbl2) $s \rightarrow_{\otimes} t$ and $s \otimes (s \rightarrow_{\otimes} t)$ are defined, imply $s \otimes (s \rightarrow_{\otimes} t) = s \wedge t$.

**Example 4.6.** Let $A = \{0, s, t, m, 1\}$, Figure 3 is the diagram of $(A; \leq)$. The operators $\otimes$ and $\rightarrow_{\otimes}$ are defined by Tables 3 and 4. Then $(A, \wedge, \vee, \otimes, \rightarrow_{\otimes}, 0, 1)$ is a partial BL-algebra.

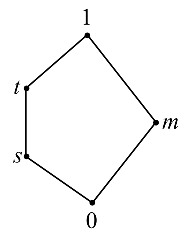


FIGURE 3. The Hasse diagram of $(A; \leq)$.

| $\otimes$ | 0 | $s$ | $t$ | $m$ | 1 |
|---|---|---|---|---|---|
| 0 | | | | | 0 |
| $s$ | | | | | $s$ |
| $t$ | | | $t$ | 0 | $t$ |
| $m$ | | | 0 | $m$ | $m$ |
| 1 | 0 | $s$ | $t$ | $m$ | 1 |

TABLE 3. Partial operation of $\otimes$.

| $\rightarrow_{\otimes}$ | 0 | $s$ | $t$ | $m$ | 1 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 |
| $s$ | | 1 | 1 | | 1 |
| $t$ | $m$ | | 1 | $m$ | 1 |
| $m$ | $t$ | | $t$ | 1 | 1 |
| 1 | 0 | $s$ | $t$ | $m$ | 1 |

TABLE 4. Partial operation of $\rightarrow_{\otimes}$.

**Theorem 4.7.** *Let $A = [0, 1]$, $\otimes$ be a cp-t-norm and $\rightarrow_{\otimes}$ be the PRI that is derived from $\otimes$. Then $(A, \wedge, \vee, \otimes, \rightarrow_{\otimes}, 0, 1)$ is a partial BL-algebra.*

*Proof.* Let $s, t \in A$.
(pbl1) Suppose $s \rightarrow_{\otimes} t$ and $t \rightarrow_{\otimes} s$ are defined.
If $t < s$, then we have $t \rightarrow_{\otimes} s = 1$. Thus $(s \rightarrow_{\otimes} t) \vee (t \rightarrow_{\otimes} s) = 1$.
If $s \leq t$, then $s \rightarrow_{\otimes} t = 1$. Thus $(s \rightarrow_{\otimes} t) \vee (t \rightarrow_{\otimes} s) = 1$.
(pbl2) Suppose $s \rightarrow_{\otimes} t$ and $s \otimes (s \rightarrow_{\otimes} t)$ are defined.
If $t < s$, then by Theorem 3.10, $s \wedge t = t = s \otimes (s \rightarrow_{\otimes} t)$.
If $s \leq t$, then $s \rightarrow_{\otimes} t = 1$. Thus $s \otimes (s \rightarrow_{\otimes} t) = s \otimes 1 = s$. So $s \wedge t = s = s \otimes (s \rightarrow_{\otimes} t)$. Hence $(A, \wedge, \vee, \otimes, \rightarrow_{\otimes}, 0, 1)$ is a partial BL-algebra obviously. □

Next, we will study a specific content partial inflationary BL-algebra in PGRL.

**Definition 4.8.** An algebra $(A, \wedge, \vee, \otimes, \rightarrow_{\otimes}, 0, 1)$ is referred to as a *partial inflationary BL-algebra* (PIBL), if it is an inflationary PGRL that fulfils: for all $s, t, m \in A$,
(pibl1) if $s \rightarrow_{\otimes} t$ and $s \otimes (s \rightarrow_{\otimes} t)$ are defined, then $s \otimes (s \rightarrow_{\otimes} t) = s \wedge t$,
(pibl2) if $s \otimes 1, t \otimes 1, s \rightarrow_{\otimes} (t \otimes 1)$ and $t \rightarrow_{\otimes} (s \otimes 1)$ are defined, then $(t \rightarrow_{\otimes} (s \otimes 1)) \vee (s \rightarrow_{\otimes} (t \otimes 1)) = 1$.

**Theorem 4.9.** *Let $A = [0, 1]$, $\otimes$ be an inflationary PO and $\rightarrow_{\otimes}$ be the PRI that is derived from $\otimes$. Then $(A, \wedge, \vee, \otimes, \rightarrow_{\otimes}, 0, 1)$ is a PIBL.*

*Proof.* It is obvious that $(A, \wedge, \vee, \otimes, \rightarrow_{\otimes}, 0, 1)$ is an inflationary PGRL. Let $s, t \in A$.

(pibl1) Since $\otimes$ is an inflationary PO, $t \rightarrow_{\otimes} s$ and $t \otimes (t \rightarrow_{\otimes} s)$ are defined and $s \leq t$. Then we get $t \otimes 0 \leq s \leq t \leq t \otimes 1$. And PO is P-continuous that there is $m \in [0, 1]$ such that $t \otimes m = s$. By the residuation principle, we get $m \leq t \rightarrow_{\otimes} s$. Thus $s = t \otimes m \leq t \otimes (t \rightarrow_{\otimes} s)$. Meanwhile, because of $t \rightarrow_{\otimes} s \leq t \rightarrow_{\otimes} s$, we obtain $t \otimes (t \rightarrow_{\otimes} s) \leq s$. So from the above, it can be concluded that $s \wedge t = s = t \otimes (t \rightarrow_{\otimes} s)$.

(pibl2) If $t \otimes 1, s \otimes 1, s \rightarrow_{\otimes} (t \otimes 1)$ and $t \rightarrow_{\otimes} (s \otimes 1)$ are defined and $s \leq t$, then it is clearly that $s \rightarrow_{\otimes} (t \otimes 1) = 1$. Thus we get $(s \rightarrow_{\otimes} (t \otimes 1)) \vee (t \rightarrow_{\otimes} (s \otimes 1)) = 1$. So $(A, \wedge, \vee, \otimes, \rightarrow_{\otimes}, 0, 1)$ is a PIBL. □

Based on the definition of algebraic structures, and the study of related properties, the algebraic structure relations can be summarized as follows:

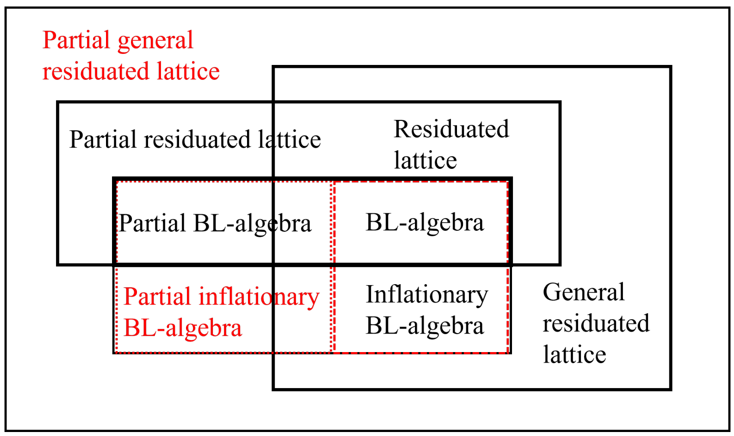


FIGURE 4. Diagram of the relations between algebraic structures such as PGRL, partial BL-algebra, etc.

## 5. Conclusions

This paper investigates the construction, properties, and residual implications of partial overlap functions (PO), partial general overlap functions (PGO), and other related operators. It examines these partial operators through the angles of algebraic structures, presenting novel concepts such as partial general residuated lattice (PGRL), partial BL-algebra, and partial inflationary BL-algebra (PIBL). These concepts are discussed in detail, with an emphasis on their interrelationships, as illustrated in Figure 4.

Future research can delve deeper into the algebraic structures introduced, particularly exploring the intricate connections between PGRL, partial BL-algebra, and non-classical logic algebras. Such studies would contribute significantly to the advancement of algebraic theory, particularly in the context of non-classical logics. Additionally, there is great potential for further investigation into the practical applications of these operations within partial fuzzy logic systems, particularly for handling incomplete information and supporting decision-making processes. Furthermore, future work could build upon existing models, such as those in [41], to extend these ideas and develop more refined approaches.

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XingNa Zhang (221711032@sust.edu.cn)
School of Mathematics and Data Science, Shaanxi University of Science and Technology, postal code 710021, Xi'an, China

Nan Sheng (snsust@163.com)
School of Mathematics and Data Science, Shaanxi University of Science and Technology, postal code 710021, Xi'an, China

Rajab Ali Borzooei (borzooei@sbu.ac.ir)
Department of Mathematics, Faculty at Mathematical Sciences, Shahid Beheshti University, postal code 1983963113, Tehran, Iran

EunSuk Yang (eunsyang@jbnu.ac.kr)
Department of Philosophy and Institute of Critical Thinking and Writing, Jeonbuk National University, Center for Humanities and Social Sciences, postal code 54896, Jeonju, Korea

XiaoHong Zhang (zhangxiaohong@sust.edu.cn)
School of Mathematics and Data Science, Shaanxi University of Science and Technology, postal code 710021, Xi'an, China